2013 – MS

Test Paper Code : MS
Time : 3 Hours Maximum Marks : 100

INSTRUCTIONS

1. This question-cum-answer booklet has 32 pages and has 30 questions. Please ensure that the copy of the question-cum-answer booklet you have received contains all the questions.

2. Write your Registration Number, Name and the name of the Test Centre in the appropriate space provided on the right side.

3. Write the answers to the objective questions against each Question No. in the Answer Table for Objective Questions, provided on Page No. 4. Do not write anything else on this page.

4. Each objective question has 4 choices for its answer: (A), (B), (C) and (D). Only ONE of them is the correct answer. There will be negative marking for wrong answers to objective questions. The following marking scheme for objective questions shall be used:
   (a) For each correct answer, you will be awarded 2 (Two) marks.
   (b) For each wrong answer, you will be awarded -0.5 (Negative 0.5) mark.
   (c) Multiple answers to a question will be treated as a wrong answer.
   (d) For each un-attempted question, you will be awarded 0 (Zero) mark.
   (e) Negative marks for objective part will be carried over to total marks.

5. Answer the fill in the blank type and descriptive type questions only in the space provided after each question. No negative marks for fill in the blank type questions.

6. Do not write more than one answer for the same question. In case you attempt a fill in the blank or a descriptive question more than once, please cancel the answer(s) you consider wrong. Otherwise the answer appearing last only will be evaluated.

7. All answers must be written in blue/black/blue-black ink only. Sketch pen, pencil or ink of any other colour should not be used.

8. All rough work should be done in the space provided and scored out finally.

9. No supplementary sheets will be provided to the candidates.

10. Clip board, log tables, slide rule, cellular phone and electronic gadgets in any form are NOT allowed. Non Programmable calculator is allowed.

11. The question-cum-answer booklet must be returned in its entirety to the Invigilator before leaving the examination hall. Do not remove any page from this booklet.

12. Refer to Special instructions/useful data on the reverse.

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<table>
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<tr>
<th>REGISTRATION NUMBER</th>
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<tr>
<td>Name:</td>
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Do not write your Registration Number or Name anywhere else in this question-cum-answer booklet.

I have read all the instructions and shall abide by them.

______________________________
Signature of the Candidate

I have verified the information filled by the Candidate above.

______________________________
Signature of the Invigilator
### Special Instructions / Useful Data

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$\mathbb{R}$</td>
<td>Set of all real numbers</td>
</tr>
<tr>
<td>$\mathbb{Q}$</td>
<td>Set of all rational numbers</td>
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<tr>
<td>$P(A)$</td>
<td>Probability of an event $A$</td>
</tr>
<tr>
<td>i.i.d.</td>
<td>independent and identically distributed</td>
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<tr>
<td>$\text{Exp}(\lambda)$</td>
<td>The exponential distribution with density $f(x; \lambda) = \begin{cases} \lambda e^{-\lambda x}, &amp; \text{if } x &gt; 0, \ 0, &amp; \text{otherwise}, \end{cases}$ for $\lambda &gt; 0$</td>
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<tr>
<td>$N(\mu, \sigma^2)$</td>
<td>Normal distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 &gt; 0$</td>
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<tr>
<td>$t_n$</td>
<td>Central Student’s t-distribution with $n$ degrees of freedom</td>
</tr>
<tr>
<td>$\chi^2_n$</td>
<td>Central Chi-square distribution with $n$ degrees of freedom</td>
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<tr>
<td>$\chi^2_{n, \alpha}$</td>
<td>a constant such that $P(W &gt; \chi^2_{n, \alpha}) = \alpha$, where $W$ has $\chi^2_n$</td>
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<tr>
<td>$\Phi(x)$</td>
<td>Cumulative distribution function of $N(0,1)$</td>
</tr>
<tr>
<td>$\varphi(x)$</td>
<td>Density function of $N(0,1)$</td>
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<tr>
<td>$A^c$</td>
<td>Complement of the event $A$</td>
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<td>$\Phi(2.33) = 0.99$</td>
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<tr>
<td>$E(X)$</td>
<td>Expectation of a random variable $X$</td>
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<td>$\text{Var}(X)$</td>
<td>Variance of a random variable $X$</td>
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<tr>
<td>$\text{Corr}(X, Y)$</td>
<td>Correlation coefficient between random variables $X$ and $Y$</td>
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<tr>
<td>$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} , dx$, $\alpha &gt; 0$</td>
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IMPORTANT NOTE FOR CANDIDATES

- Questions 1-10 (objective questions) carry two marks each, questions 11-20 (fill in the blank questions) carry three marks each and questions 21-30 (descriptive questions) carry five marks each.
- The marking scheme for the objective type question, is as follows:
  (a) For each correct answer, you will be awarded 2 (Two) marks.
  (b) For each wrong answer, you will be awarded -0.5 (Negative 0.5) mark.
  (c) Multiple answers to a question will be treated as a wrong answer.
  (d) For each un-attempted question, you will be awarded 0 (Zero) mark.
  (e) Negative marks for objective part will be carried over to total marks.
- There is no negative marking for fill in the blank questions.
- Write the answers to the objective questions in the Answer Table for Objective Questions provided on page 4 only.

Objective Questions

Q.1 Let $E$ and $F$ be two events with $P(E) = 0.7, P(F) = 0.4$ and $P(E \cap F') = 0.4$. Then $P(F | E \cup F')$ is equal to

(A) $\frac{1}{2}$  (B) $\frac{1}{3}$  (C) $\frac{1}{4}$  (D) $\frac{1}{5}$

Q.2 Let $\{a_n\}_{n \geq 1}$ be a sequence of positive real numbers such that $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \frac{1}{2}$. Then $\lim_{n \to \infty} \frac{a_n^2 + a_n}{4}$ is equal to

(A) $\infty$  (B) $\frac{e^{\sqrt{4}}}{4} + \frac{1}{8}$  (C) $\frac{e^{\sqrt{4}}}{4}$  (D) $\frac{1}{4}$

Q.3 Let $f : [0, \infty) \to [0, \infty)$ be a twice differentiable and increasing function with $f(0) = 0$. Suppose that, for any $t \geq 0$, the length of the arc of the curve $y = f(x)$, $x \geq 0$ between $x = 0$ and $x = t$ is $\frac{2}{3} \left[ (1+t)^{3/2} - 1 \right]$. Then $f(4)$ is equal to

(A) $\frac{11}{3}$  (B) $\frac{13}{3}$  (C) $\frac{14}{3}$  (D) $\frac{16}{3}$
Q.4 Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be defined by
\[
f(x, y) = \begin{cases} 
\frac{\sin(2(x^2 + y^2))}{x^2 + y^2} e^{3x \sin\left(\frac{y}{x}\right)} & \text{if } (x, y) \neq (0, 0), \\
\alpha & \text{if } (x, y) = (0, 0),
\end{cases}
\]
where \( \alpha \) is a real constant. If \( f \) is continuous at \((0,0)\), then \( \alpha \) is equal to
(A) 1  
(B) 2  
(C) 3  
(D) 4

Q.5 Let \( A \) be a \( 3 \times 3 \) real matrix with eigenvalues 1, 2, 3 and let \( B = A^{-1} + A^2 \). Then the trace of the matrix \( B \) is equal to
(A) \( \frac{91}{6} \)  
(B) \( \frac{95}{6} \)  
(C) \( \frac{97}{6} \)  
(D) \( \frac{101}{6} \)

Q.6 Let \( X_1, X_2, \ldots \) be a sequence of i.i.d. random variables with variance 1. Then
\[
\lim_{n \to \infty} \frac{(X_1 - X_2) + (X_3 - X_4) + \cdots + (X_{2n} - X_{2n})}{\sqrt{n}} \leq x
\]
is equal to
(A) \( \Phi(x) \)  
(B) \( \Phi(2x) \)  
(C) \( \Phi\left(x\sqrt{2}\right) \)  
(D) \( \Phi\left(\frac{x}{\sqrt{2}}\right) \)

Q.7 Let \( X_1, X_2, \ldots, X_{100} \) be a random sample from a \( N(2,4) \) population. Let
\[
\bar{X} = \frac{1}{99} \sum_{i=1}^{99} X_i, \quad S = \sqrt{\frac{1}{98} \sum_{i=1}^{99} (X_i - \bar{X})^2} \quad \text{and} \quad W = \frac{X_{100} - 2}{S}.
\]
Then the distribution of \( W \) is
(A) \( X_{98}^2 \)  
(B) \( X_{99}^2 \)  
(C) \( t_{98} \)  
(D) \( t_{99} \)

Q.8 Let \( X_1, X_2, \ldots, X_n, X_{n+1} \) be a random sample from a \( N(\mu,1) \) population. If \( \bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \) and
\[
T = \frac{1}{2} (\bar{X} + X_{n+1}),
\]
then for estimating \( \mu \)
(A) \( T \) is unbiased and consistent  
(B) \( T \) is biased and consistent  
(C) \( T \) is unbiased and inconsistent  
(D) \( T \) is biased and inconsistent
Q.9 Let $X$ be an observation from a population with density

$$f(x) = \begin{cases} \lambda^2 x e^{-\lambda x}, & \text{if } x > 0, \lambda > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

For testing $H_0 : \lambda = 2$ against $H_1 : \lambda = 1$, the most powerful test of size $\alpha$ is given by “Reject $H_0$ if $X > c$”, where $c$ is given by

(A) $\frac{1}{4} \chi^2_{4, \alpha}$  
(B) $\frac{1}{4} \chi^2_{3, \alpha}$  
(C) $\frac{1}{4} \chi^2_{2, \alpha}$  
(D) $\frac{1}{4} \chi^2_{1, \alpha}$

Q.10 A continuous random variable $X$ has the density

$$f(x) = 2 \varphi(x) \Phi(x), \ x \in \mathbb{R}.$$ 

Then

(A) $E(X) > 0$  
(B) $E(X) < 0$  
(C) $P(X \leq 0) > 0.5$  
(D) $P(X \geq 0) < 0.25$
## Answer Table for Objective Questions

Write the Code of your chosen answer only in the 'Answer' column against each Question Number. Do not write anything else on this page.

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<th>Question Number</th>
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**FOR EVALUATION ONLY**

<table>
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<td>Number of Incorrect Answers</td>
<td>Marks</td>
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<tr>
<td><strong>Total Marks in Questions 1-10</strong></td>
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</table>
Fill in the blank questions

Q.11 If $X$ has the probability density function

\[ f(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}; \quad x > 0, \alpha > 2, \]

then $\text{Var}\left( \frac{1}{X} \right)$ is equal to

Ans:

Q.12 Let the joint density function of $(X, Y)$ be

\[ f(x, y) = \begin{cases} 
  c (x+y), & \text{if } -x < y < x, \quad 0 < x < 1, \\
  0, & \text{otherwise}.
\end{cases} \]

Then the value of $c$ is equal to

Ans:

Q.13 Let $X$ be an observation from a population with density function $f(x)$. Then the power of the most powerful test of size $\alpha = 0.19$ for testing

\[ H_0: f(x) = \begin{cases} 
  \frac{x}{2}, & \text{if } 0 < x < 2, \\
  0, & \text{otherwise},
\end{cases} \]

against $H_1: f(x) = \begin{cases} 
  \frac{3x^2}{8}, & \text{if } 0 < x < 2, \\
  0, & \text{otherwise},
\end{cases}$

is equal to

Ans:

Q.14 Bulbs produced by a factory $F_i$ have lifetimes (in months) distributed as $\text{Exp}\left( \frac{1}{3^i} \right)$ for $i=1,2,3$. A firm randomly procures 40% of its required bulbs from $F_1$, 30% from $F_2$ and 30% from $F_3$. A randomly selected bulb from the firm is found to be working after 27 months. The probability that it was produced by the factory $F_3$ is

Ans:

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Q.15 Let $X_1, \ldots, X_n$ be a random sample from a population with density

$$f(x, \mu) = \begin{cases} e^{\mu-x}, & \text{if } x > \mu, \\ 0, & \text{otherwise,} \end{cases}$$

and let $X_{(i)} = \min\{X_1, X_2, \ldots, X_n\}$. Then $\left(X_{(i)} - \frac{2}{n} \log e 5, X_{(i)}\right)$ is a $\ldots\ldots\ldots\%$ confidence interval for $\mu$.

Ans:

---

Q.16 Ten percent of bolts produced in a factory are defective. They are randomly packed in boxes such that each box contains 3 bolts. Four of these boxes are bought by a customer. The probability, that the boxes that this customer bought have no defective bolt in them, is equal to

Ans:

---

Q.17 Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2, & \text{if } x \in \mathbb{Q}, \\ \alpha x + \beta, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

where $\alpha$ and $\beta$ are real constants. If $f$ is differentiable at $x = 1$ then the value of $3\alpha + \beta$ is equal to

Ans:

---

Q.18 Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $|a_n| \leq \sqrt{n}$, $n = 1, 2, \ldots$. Then

$$\lim_{n \to \infty} \left\{ e^{a_n} + \sqrt{n} \sin \left( \frac{2}{\sqrt{n}} \right) \right\}$$

is equal to

Ans:

---
Q.19 Consider the linear system

\begin{align*}
    x + y + 2z &= \alpha \\
    x + 4y + z &= 4 \\
    3y - z &= \gamma 
\end{align*}

in the unknowns \(x, y\) and \(z\). If the above system always has a solution then the value of \(\alpha + \gamma\) is equal to

Ans:

Q.20 The general solution of the differential equation \((x^4 - y) \, dx + (y^4 - x) \, dy = 0\) is equal to

Ans:
Descriptive questions

Q.21 Consider the matrix

\[ P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & \alpha & \beta \\ \alpha & 0 & \beta \end{bmatrix} \]

If \( P \) has eigenvalues 0 and 3 then determine the values of the pair \((\alpha, \beta)\).
Q.22 Let a function $f: [0,1] \to \mathbb{R}$ be continuous on $[0,1]$ and differentiable in $(0,1)$. If $f(0) = 1$ and $[f(1)]^3 + 2f(1) = 5$, then prove that there exists a $c \in (0,1)$ such that $f'(c) = \frac{2}{2 + 3[f(c)]^2}$.
Q.23 Let \( \{a_n\}_{n \geq 1} \) be a sequence of real numbers such that \( \sum_{n=1}^{\infty} a_n \) converges absolutely. Prove that the series \( \sum_{n=1}^{\infty} \log_e (1 + a_n^4) \) converges.
Q.24 Let \( D = \{ (x, y) \in \mathbb{R}^2 : 0 \leq y \leq x \leq 1 \} \) and let \( f : D \to \mathbb{R} \) be defined by
\[
f(x, y) = x^2 - 2xy + 2, \quad (x, y) \in D.
\]
Then determine the maximum value of \( f \) in the region \( D \).
Q.25 Let $X, Y$ and $Z$ be independent random variables with respective moment generating functions 

$$M_X(t) = \frac{1}{1-t}, \quad t < 1; \quad M_Y(t) = e^{\frac{t}{2}} = M_Z(t), \quad t \in \mathbb{R}.$$ 

Let $W = 2X + Y^2 + Z^2$. Then determine the value of $P(W > 2)$. 
Q.26 Ram rolls a pair of fair dice. If the sum of the numbers shown on the upper faces is 5, 6, 10, 11 or 12 then Ram wins a gold coin. Otherwise, he rolls the pair of dice once again and wins a silver coin if the sum of the numbers shown on the upper faces in the second throw is the same as the sum of the numbers in the first throw. What is the probability that he wins a gold or a silver coin?
Q.27 Let $X_1, \ldots, X_n$ be a random sample from a uniform distribution on the interval $[\theta, 2\theta], \; \theta > 0$. Find the method of moments estimator and the maximum likelihood estimator of $\theta$. Further find the bias of the maximum likelihood estimator.
Q.28 Let \((X_1, Y_1), (X_2, Y_2), \ldots\) be a sequence of i.i.d. bivariate normal random variables with 
\[ E(X_1) = 75, \quad E(Y_1) = 25, \quad \text{Var}(X_1) = 36, \quad \text{Var}(Y_1) = 16 \quad \text{and} \quad \text{Corr}(X_1, Y_1) = 0.25. \]
Let 
\[ \bar{U} = \frac{1}{n} \sum_{i=1}^{n} (X_i + Y_i). \]
Find the minimum value of \(n\) so that \(P(\bar{U} \leq 104) \geq 0.99.\)
Q.29 The joint probability density function of \((X, Y)\) is

\[
f(x, y) = \begin{cases} 
\frac{1}{2} e^{(x-y)}, & \text{if } x+y > 1, \ x > 0, \ y > 0, \\
0, & \text{otherwise.}
\end{cases}
\]

Find the probability density function of \(X\) and \(E(Y \mid X = x), \ x > 0.\)
Q.30 Suppose that $F$ is a cumulative distribution function, where

$$F(x) = \begin{cases} 
0, & \text{if } x < 0, \\
1 - e^{-x}, & \text{if } 0 \leq x < 1, \\
c, & \text{if } 1 \leq x < 2, \\
1 - e^{-x}, & \text{if } x \geq 2.
\end{cases}$$

i. Find all possible values of $c$.

ii. Find $P(0.5 \leq X \leq 2.5)$ and $P(X = 1) + P(X = 2)$. 

Space for the answer
Space for rough work

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# 2013 - MS

## Objective Part

(Question Number 1 – 10)

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## Fill in the blanks Part and Descriptive Part

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Total Marks in Fill in the blanks Part and Descriptive Part

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