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Φ(0.25) = 0.5987, Φ(0.5) = 0.6915, Φ(0.625) = 0.7341, Φ(0.71) = 0.7612, Φ(1) = 0.8413, Φ(1.125) = 0.8697, Φ(1.5) = 0.9332, Φ(1.64) = 0.95, Φ(2) = 0.9772
SECTION – A
MULTIPLE CHOICE QUESTIONS (MCQ)

Q. 1 – Q.10 carry one mark each.

Q.1 The imaginary parts of the eigenvalues of the matrix

\[ P = \begin{pmatrix} 3 & 2 & 5 \\ 2 & -3 & 6 \\ 0 & 0 & -3 \end{pmatrix} \]

are

(A) 0, 0, 0  (B) 2, -2, 0  (C) 1, -1, 0  (D) 3, -3, 0

Q.2 Let \( u, v \in \mathbb{R}^4 \) be such that \( u = (1 \quad 2 \quad 3 \quad 5)^T \) and \( v = (5 \quad 3 \quad 2 \quad 1)^T \). Then the equation \( uu^T x = v \) has

(A) infinitely many solutions  (B) no solution  
(C) exactly one solution  (D) exactly two solutions

Q.3 Let \( u_n = \left( 4 - \frac{1}{n} \right)^{(-1)n}, n \in \mathbb{N} \) and let \( l = \lim_{n \to \infty} u_n \).

Which of the following statements is TRUE?

(A) \( l = 0 \) and \( \sum_{n=1}^{\infty} u_n \) is convergent
(B) \( l = \frac{1}{4} \) and \( \sum_{n=1}^{\infty} u_n \) is divergent
(C) \( l = \frac{1}{4} \) and \( \{u_n\}_{n \geq 1} \) is oscillatory
(D) \( l = 1 \) and \( \sum_{n=1}^{\infty} u_n \) is divergent

Q.4 Let \( \{a_n\}_{n \geq 1} \) be a sequence defined as follows:

\[ a_1 = 1 \text{ and } a_{n+1} = \frac{7a_n + 11}{21}, n \in \mathbb{N}. \]

Which of the following statements is TRUE?

(A) \( \{a_n\}_{n \geq 1} \) is an increasing sequence which diverges
(B) \( \{a_n\}_{n \geq 1} \) is an increasing sequence with \( \lim_{n \to \infty} a_n = \frac{11}{14} \)
(C) \( \{a_n\}_{n \geq 1} \) is a decreasing sequence which diverges
(D) \( \{a_n\}_{n \geq 1} \) is a decreasing sequence with \( \lim_{n \to \infty} a_n = \frac{11}{14} \)
Q.5 Let $X$ be a continuous random variable with the probability density function

$$f(x) = \begin{cases} 
0, & \text{if } x \leq 0 \\
x^3, & \text{if } 0 < x \leq 1 \\
\frac{3}{x^5}, & \text{if } x > 1 
\end{cases}$$

Then $P\left(\frac{1}{2} < X < 2\right)$ equals

(A) $\frac{15}{16}$ \quad (B) $\frac{11}{16}$ \quad (C) $\frac{7}{12}$ \quad (D) $\frac{3}{8}$

Q.6 Let $X$ be a random variable with the moment generating function

$$M_X(t) = \frac{1}{216}(5 + e^t)^3, \quad t \in \mathbb{R}.$$ 

Then $P(X > 1)$ equals

(A) $\frac{2}{27}$ \quad (B) $\frac{1}{27}$ \quad (C) $\frac{1}{12}$ \quad (D) $\frac{2}{9}$

Q.7 Let $X$ be a discrete random variable with the probability mass function

$$p(x) = k(1 + |x|)^2, \quad x = -2, -1, 0, 1, 2,$$

where $k$ is a real constant. Then $P(X = 0)$ equals

(A) $\frac{1}{9}$ \quad (B) $\frac{2}{27}$ \quad (C) $\frac{1}{27}$ \quad (D) $\frac{1}{81}$

Q.8 Let the random variable $X$ have uniform distribution on the interval $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$. Then $P(\cos X > \sin X)$ is

(A) $\frac{2}{3}$ \quad (B) $\frac{1}{2}$ \quad (C) $\frac{1}{3}$ \quad (D) $\frac{1}{4}$

Q.9 Let $\{X_n\}_{n \geq 1}$ be a sequence of i.i.d. random variables having common probability density function

$$f(x) = \begin{cases} 
xe^{-x}, & \text{if } x \geq 0 \\
0, & \text{otherwise} 
\end{cases}$$

Let $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, $n = 1, 2, \ldots$. Then $\lim_{n \to \infty} P(\overline{X}_n = 2)$ equals

(A) 0 \quad (B) $\frac{1}{4}$ \quad (C) $\frac{1}{2}$ \quad (D) 1
Q.10 Let \( X_1, X_2, X_3 \) be a random sample from a distribution with the probability density function
\[
f(x|\theta) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}}, & \text{if } x > 0, \\ 0, & \text{otherwise} \end{cases}, \quad \theta > 0.
\]
Which of the following estimators of \( \theta \) has the smallest variance for all \( \theta > 0 \)?
(A) \( \frac{X_1 + 3X_2 + X_3}{5} \)
(B) \( \frac{X_1 + X_2 + 2X_3}{4} \)
(C) \( \frac{X_1 + X_2 + X_3}{3} \)
(D) \( \frac{X_1 + 2X_2 + 3X_3}{6} \)

Q. 11 – Q. 30 carry two marks each.

Q.11 Player \( P_1 \) tosses 4 fair coins and player \( P_2 \) tosses a fair die independently of \( P_1 \). The probability that the number of heads observed is more than the number on the upper face of the die, equals
(A) \( \frac{7}{16} \)
(B) \( \frac{5}{32} \)
(C) \( \frac{17}{96} \)
(D) \( \frac{21}{64} \)

Q.12 Let \( X_1 \) and \( X_2 \) be i.i.d. continuous random variables with the probability density function
\[
f(x) = \begin{cases} 6x(1-x), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}.
\]
Using Chebyshev’s inequality, the lower bound of \( P \left( |X_1 + X_2 - 1| \leq \frac{1}{2} \right) \) is
(A) \( \frac{5}{6} \)
(B) \( \frac{4}{5} \)
(C) \( \frac{3}{5} \)
(D) \( \frac{1}{3} \)

Q.13 Let \( X_1, X_2, X_3 \) be i.i.d. discrete random variables with the probability mass function
\[
p(k) = \left( \frac{2}{3} \right)^{k-1} \left( \frac{1}{3} \right), \quad k = 1, 2, 3, ...
\]
Let \( Y = X_1 + X_2 + X_3 \). Then \( P(Y \geq 5) \) equals
(A) \( \frac{1}{9} \)
(B) \( \frac{8}{9} \)
(C) \( \frac{2}{27} \)
(D) \( \frac{25}{27} \)

Q.14 Let \( X \) and \( Y \) be continuous random variables with the joint probability density function
\[
f(x, y) = \begin{cases} cx(1-x), & \text{if } 0 < x < y < 1 \\ 0, & \text{otherwise} \end{cases},
\]
where \( c \) is a positive real constant. Then \( E(X) \) equals
(A) \( \frac{1}{5} \)
(B) \( \frac{1}{4} \)
(C) \( \frac{2}{5} \)
(D) \( \frac{1}{3} \)
Q.15 Let $X$ and $Y$ be continuous random variables with the joint probability density function

$$f(x, y) = \begin{cases} x + y, & \text{if } 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}.$$

Then $P \left( X + Y > \frac{3}{2} \right)$ equals

(A) $\frac{23}{24}$ (B) $\frac{1}{12}$ (C) $\frac{11}{12}$ (D) $\frac{1}{24}$

Q.16 Let $X_1, X_2, \ldots, X_m, Y_1, Y_2, \ldots, Y_n$ be i.i.d. $N(0, 1)$ random variables. Then

$$W = \frac{m \left( \sum_{i=1}^{m} X_i \right)^2}{m \left( \sum_{i=1}^{n} Y_i \right)^2}$$

has

(A) $\chi^2_{m+n}$ distribution (B) $t_n$ distribution (C) $F_{m,n}$ distribution (D) $F_1,n$ distribution

Q.17 Let $(X_n)_{n \geq 1}$ be a sequence of i.i.d. random variables with the probability mass function

$$f(x) = \begin{cases} \frac{1}{4}, & \text{if } x = 4 \\ \frac{1}{4}, & \text{if } x = 8 \\ 0, & \text{otherwise} \end{cases}$$

Let $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$, $n = 1, 2, \ldots$. If $\lim_{n \to \infty} P(m \leq \bar{X}_n \leq M) = 1$, then possible values of $m$ and $M$ are

(A) $m = 2.1$, $M = 3.1$ (B) $m = 3.2$, $M = 4.1$

(C) $m = 4.2$, $M = 5.7$ (D) $m = 6.1$, $M = 7.1$

Q.18 Let $x_1 = 1.1$, $x_2 = 0.5$, $x_3 = 1.4$, $x_4 = 1.2$ be the observed values of a random sample of size four from a distribution with the probability density function

$$f(x|\theta) = \begin{cases} e^{\theta - x}, & \text{if } x \geq \theta \\ 0, & \text{otherwise} \end{cases}, \quad \theta \in (-\infty, \infty).$$

Then the maximum likelihood estimate of $\theta^2$ is

(A) 0.5 (B) 0.25 (C) 1.21 (D) 1.44
Let $x_1 = 2$, $x_2 = 1$, $x_3 = \sqrt{5}$, $x_4 = \sqrt{2}$ be the observed values of a random sample of size four from a distribution with the probability density function

\[ f(x|\theta) = \begin{cases} \frac{1}{2\theta}, & \text{if } -\theta \leq x \leq \theta, \\ 0, & \text{otherwise} \end{cases}, \quad \theta > 0. \]

Then the method of moments estimate of $\theta$ is

(A) 1 \hspace{1cm} (B) 2 \hspace{1cm} (C) 3 \hspace{1cm} (D) 4

Let $X_1, X_2$ be a random sample from an $N(0, \theta)$ distribution, where $\theta > 0$. Then the value of $k$, for which the interval $(0, \frac{x_1^2 + x_2^2}{k})$ is a 95% confidence interval for $\theta$, equals

(A) $-\log_e(0.95)$ \hspace{1cm} (B) $-2 \log_e(0.95)$ \hspace{1cm} (C) $-\frac{1}{2} \log_e(0.95)$ \hspace{1cm} (D) 2

Let $X_1, X_2, X_3, X_4$ be a random sample from $N(\theta_1, \sigma^2)$ distribution and $Y_1, Y_2, Y_3, Y_4$ be a random sample from $N(\theta_2, \sigma^2)$ distribution, where $\theta_1, \theta_2 \in (-\infty, \infty)$ and $\sigma > 0$. Further suppose that the two random samples are independent. For testing the null hypothesis $H_0: \theta_1 = \theta_2$ against the alternative hypothesis $H_1: \theta_1 > \theta_2$, suppose that a test $\psi$ rejects $H_0$ if and only if $\Sigma_{i=1}^4 X_i > \Sigma_{j=1}^4 Y_j$. The power of the test $\psi$ at $\theta_1 = 1 + \sqrt{2}$, $\theta_2 = 1$ and $\sigma^2 = 4$ is

(A) 0.5987 \hspace{1cm} (B) 0.7341 \hspace{1cm} (C) 0.7612 \hspace{1cm} (D) 0.8413

Let $X$ be a random variable having a probability density function $f \in \{f_0, f_1\}$, where

\[ f_0(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases} \]

and

\[ f_1(x) = \begin{cases} \frac{1}{2}, & \text{if } 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases} \]

For testing the null hypothesis $H_0: f \equiv f_0$ against $H_1: f \equiv f_1$, based on a single observation on $X$, the power of the most powerful test of size $\alpha = 0.05$ equals

(A) 0.425 \hspace{1cm} (B) 0.525 \hspace{1cm} (C) 0.625 \hspace{1cm} (D) 0.725

If

\[ \int_{y=0}^{1} \int_{x=y}^{2-\sqrt{1-(y-1)^2}} f(x,y) \, dx \, dy = \int_{x=0}^{1} \int_{y=0}^{\alpha(x)} f(x,y) \, dy \, dx + \int_{x=1}^{2} \int_{y=0}^{\beta(x)} f(x,y) \, dy \, dx, \]

then $\alpha(x)$ and $\beta(x)$ are

(A) $\alpha(x) = x$, $\beta(x) = 1 + \sqrt{1-(x-2)^2}$ \hspace{1cm} (B) $\alpha(x) = x$, $\beta(x) = 1 - \sqrt{1-(x-2)^2}$

(C) $\alpha(x) = 1 + \sqrt{1-(x-2)^2}$, $\beta(x) = x$ \hspace{1cm} (D) $\alpha(x) = 1 - \sqrt{1-(x-2)^2}$, $\beta(x) = x$
Q.24 Let $f: [0,1] \rightarrow \mathbb{R}$ be a function defined as

$$f(t) = \begin{cases} t^3 \left( 1 + \frac{1}{5} \cos(\log_e t^4) \right) & \text{if } t \in (0,1) \\ 0 & \text{if } t = 0 \end{cases}.$$ 

Let $F: [0,1] \rightarrow \mathbb{R}$ be defined as

$$F(x) = \int_0^x f(t) \, dt.$$ 

Then $F''(0)$ equals

(A) 0 \quad (B) \frac{3}{5} \quad (C) -\frac{5}{3} \quad (D) \frac{1}{5}$

Q.25 Consider the function

$$f(x, y) = x^3 - y^3 - 3x^2 + 3y^2 + 7, \; x, y \in \mathbb{R}.$$ 

Then the local minimum (m) and the local maximum (M) of $f$ are given by

(A) $m = 3, \; M = 7$ \quad (B) $m = 4, \; M = 11$
(C) $m = 7, \; M = 11$ \quad (D) $m = 3, \; M = 11$

Q.26 For $c \in \mathbb{R}$, let the sequence $\{u_n\}_{n=1}^{\infty}$ be defined by

$$u_n = \frac{(1 + \frac{c}{n})^{n^2}}{(3 - \frac{1}{n})^n}.$$ 

Then the values of $c$ for which the series $\sum_{n=1}^{\infty} u_n$ converges are

(A) $\log_e 6 < c < \log_e 9$ \quad (B) $c < \log_e 3$
(C) $\log_e 9 < c < \log_e 12$ \quad (D) $\log_e 3 < c < \log_e 6$

Q.27 If for a suitable $\alpha > 0$,

$$\lim_{x \to 0} \left( \frac{1}{e^{2x} - 1} - \frac{1}{\alpha x} \right)$$

exists and is equal to $l$ ($|l| < \infty$), then

(A) $\alpha = 2, \; l = 2$ \quad (B) $\alpha = 2, \; l = -\frac{1}{2}$
(C) $\alpha = \frac{1}{2}, \; l = -2$ \quad (D) $\alpha = \frac{1}{2}, \; l = \frac{1}{2}$
Q.28 Let

\[ P = \int_{0}^{1} \frac{dx}{\sqrt{8 - x^2 - x^3}}. \]

Which of the following statements is TRUE?

(A) \( \sin^{-1} \left( \frac{1}{\sqrt{2}} \right) < P < \frac{1}{\sqrt{2}} \sin^{-1} \left( \frac{1}{2} \right) \)

(B) \( \frac{1}{\sqrt{2}} \sin^{-1} \left( \frac{1}{2} \right) < P < \sin^{-1} \left( \frac{1}{2} \right) \)

(C) \( \frac{1}{\sqrt{2}} \sin^{-1} \left( \frac{1}{2} \right) < P < \sin^{-1} \left( \frac{1}{\sqrt{2}} \right) \)

(D) \( \sin^{-1} \left( \frac{1}{2} \right) < P < \frac{\sqrt{3}}{2} \sin^{-1} \left( \frac{1}{2} \right) \)

Q.29 Let \( Q, A, B \) be matrices of order \( n \times n \) with real entries such that \( Q \) is orthogonal and \( A \) is invertible. Then the eigenvalues of \( Q^T A^{-1} B Q \) are always the same as those of

(A) \( AB \)

(B) \( Q^T A^{-1} B \)

(C) \( A^{-1} B Q^T \)

(D) \( BA^{-1} \)

Q.30 Let \( (x(t), y(t)), 1 \leq t \leq \pi, \) be the curve defined by

\[ x(t) = \int_{1}^{t} \frac{\cos z}{z^2} \, dz \quad \text{and} \quad y(t) = \int_{1}^{t} \frac{\sin z}{z^2} \, dz. \]

Let \( L \) be the length of the arc of this curve from the origin to the point \( P \) on the curve at which the tangent is perpendicular to the \( x \)-axis. Then \( L \) equals

(A) \( \sqrt{2} \)

(B) \( \frac{\pi}{\sqrt{2}} \)

(C) \( 1 - \frac{2}{\pi} \)

(D) \( \frac{\pi}{2} + \sqrt{2} \)

SECTION - B

MULTIPLE SELECT QUESTIONS (MSQ)

Q. 31 – Q. 40 carry two marks each.

Q.31 Let \( v \in \mathbb{R}^k \) with \( v^T v \neq 0 \). Let

\[ P = I - 2 \frac{vv^T}{v^T v}, \]

where \( I \) is the \( k \times k \) identity matrix. Then which of the following statements is (are) TRUE?

(A) \( P^{-1} = I - P \)

(B) \( -1 \) and \( 1 \) are eigenvalues of \( P \)

(C) \( P^{-1} = P \)

(D) \( (I + P)v = v \)
Q.32 Let \( \{a_n\}_{n=1}^{\infty} \) and \( \{b_n\}_{n=1}^{\infty} \) be sequences of real numbers such that \( \{a_n\}_{n=1}^{\infty} \) is increasing and \( \{b_n\}_{n=1}^{\infty} \) is decreasing. Under which of the following conditions, the sequence \( \{a_n + b_n\}_{n=1}^{\infty} \) is always convergent?

(A) \( \{a_n\}_{n=1}^{\infty} \) and \( \{b_n\}_{n=1}^{\infty} \) are bounded sequences

(B) \( \{a_n\}_{n=1}^{\infty} \) is bounded above

(C) \( \{a_n\}_{n=1}^{\infty} \) is bounded above and \( \{b_n\}_{n=1}^{\infty} \) is bounded below

(D) \( a_n \to \infty \) and \( b_n \to -\infty \)

Q.33 Let \( f: [0,1] \to [0,1] \) be defined as follows:

\[
f(x) = \begin{cases} 
  x, & \text{if } x \in \mathbb{Q} \cap [0,1] \\
  x + \frac{2}{3}, & \text{if } x \in \mathbb{Q}^c \cap \left(0, \frac{1}{3}\right) \\
  x - \frac{1}{3}, & \text{if } x \in \mathbb{Q}^c \cap \left(\frac{1}{3}, 1\right)
\end{cases}
\]

Which of the following statements is (are) TRUE?

(A) \( f \) is one-one and onto

(B) \( f \) is not one-one but onto

(C) \( f \) is continuous on \( \mathbb{Q} \cap [0,1] \)

(D) \( f \) is discontinuous everywhere on \( [0,1] \)

Q.34 Let \( f(x) \) be a nonnegative differentiable function on \([a, b] \subseteq \mathbb{R}\) such that \( f(a) = 0 = f(b) \) and \(|f'(x)| \leq 4\). Let \( L_1 \) and \( L_2 \) be the straight lines given by the equations \( y = 4(x - a) \) and \( y = -4(x - b) \), respectively. Then which of the following statements is (are) TRUE?

(A) The curve \( y = f(x) \) will always lie below the lines \( L_1 \) and \( L_2 \)

(B) The curve \( y = f(x) \) will always lie above the lines \( L_1 \) and \( L_2 \)

(C) \( \int_a^b f(x) \, dx < (b - a)^2 \)

(D) The point of intersection of the lines \( L_1 \) and \( L_2 \) lie on the curve \( y = f(x) \)

Q.35 Let \( E \) and \( F \) be two events with \( 0 < P(E) < 1, \ 0 < P(F) < 1 \) and \( P(E) + P(F) \geq 1 \). Which of the following statements is (are) TRUE?

(A) \( P(E^C) \leq P(F) \)

(B) \( P(E \cup F) < P(E^C \cup F^C) \)

(C) \( P(E|F^C) \geq P(F^C|E) \)

(D) \( P(E^C|F) \leq P(F|E^C) \)
Q.36  The cumulative distribution function of a random variable $X$ is given by

$$F(x) = \begin{cases} 
0, & \text{if } x < 0 \\
\frac{4}{9}, & \text{if } 0 \leq x < 1 \\
\frac{8}{9}, & \text{if } 1 \leq x < 2 \\
1, & \text{if } x \geq 2 
\end{cases}$$

Which of the following statements is (are) TRUE?

(A) The random variable $X$ takes positive probability only at two points

(B) $P(1 \leq X \leq 2) = \frac{5}{9}$

(C) $E(X) = \frac{2}{3}$

(D) $P(0 < X < 1) = \frac{4}{9}$

Q.37  Let $X_1, X_2$ be a random sample from a distribution with the probability mass function

$$f(x|\theta) = \begin{cases} 
1 - \theta, & \text{if } x = 0 \\
\theta, & \text{if } x = 1 \\
0, & \text{otherwise}
\end{cases} \quad 0 < \theta < 1.$$ 

Which of the following is (are) unbiased estimator(s) of $\theta$?

(A) $\frac{X_1 + X_2}{2}$

(B) $\frac{X_1^2 + X_2^2}{2}$

(C) $\frac{X_1^2 + X_2^2}{2}$

(D) $\frac{X_1 + X_2 - X_1^2}{2}$

Q.38  Let $X_1, X_2, X_3$ be a random sample from a distribution with the probability density function

$$f(x|\theta) = \begin{cases} 
\frac{1}{\theta} e^{-x/\theta}, & \text{if } x > 0 \\
0, & \text{otherwise}
\end{cases} \quad \theta > 0.$$ 

If $\bar{\delta}(X_1, X_2, X_3)$ is an unbiased estimator of $\theta$, which of the following CANNOT be attained as a value of the variance of $\bar{\delta}$ at $\theta = 1$?

(A) 0.1

(B) 0.2

(C) 0.3

(D) 0.5

Q.39  Let $X_1, X_2, \ldots, X_n$ $(n \geq 2)$ be a random sample from a distribution with the probability density function

$$f(x|\theta) = \begin{cases} 
\frac{x}{\theta^2} e^{-x/\theta}, & \text{if } x > 0 \\
0, & \text{otherwise}
\end{cases} \quad \theta > 0.$$ 

Let $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$. Which of the following statistics is (are) sufficient but NOT complete?

(A) $\bar{X}$

(B) $\bar{X}^2 + 3$

(C) $(X_1, \sum_{i=2}^{n} X_i)$

(D) $(X_1, \bar{X})$
Q.40 Let \( X_1, X_2, X_3, X_4 \) be a random sample from an \( N(\theta, 1) \) distribution, where \( \theta \in (-\infty, \infty) \). Suppose the null hypothesis \( H_0: \theta = 1 \) is to be tested against the hypothesis \( H_1: \theta < 1 \) at \( \alpha = 0.05 \) level of significance. For what observed values of \( \sum_{i=1}^{4} X_i \), the uniformly most powerful test would reject \( H_0 \)?

(A) \(-1\) \hspace{1cm} (B) \(0\) \hspace{1cm} (C) \(0.5\) \hspace{1cm} (D) \(0.8\)

SECTION – C

NUMERICAL ANSWER TYPE (NAT)

Q. 41 – Q. 50 carry one mark each.

Q.41 Let the random variable \( X \) have uniform distribution on the interval \((0, 1)\) and \( Y = -2 \log_ e X \). Then \( E(Y) \) equals ________

Q.42 If \( Y = \log_{10} X \) has \( N(\mu, \sigma^2) \) distribution with moment generating function \( M_Y(t) = e^{st+2t^2}, t \in (-\infty, \infty) \), then \( P(X < 1000) \) equals __________

Q.43 Let \( X_1, X_2, X_3, X_4, X_5 \) be independent random variables with \( X_1 \sim N(200, 8) \), \( X_2 \sim N(104, 8) \), \( X_3 \sim N(108, 15) \), \( X_4 \sim N(120, 15) \) and \( X_5 \sim N(210, 15) \). Let \( U = \frac{X_1+X_2}{2} \) and \( V = \frac{X_3+X_4+X_5}{3} \). Then \( P(U > V) \) equals __________

Q.44 Let \( X \) and \( Y \) be discrete random variables with the joint probability mass function

\[
p(x, y) = \frac{1}{25} (x^2 + y^2), \text{ if } x = 1, 2; y = 0, 1, 2.
\]

Then \( P(Y = 1 | X = 1) \) equals __________

Q.45 Let \( X \) and \( Y \) be continuous random variables with the joint probability density function

\[
f(x, y) = \begin{cases} 8xy, & 0 < y < x < 1 \\ 0, & \text{otherwise} \end{cases}
\]

Then \( 9 \text{Cov}(X, Y) \) equals __________

Q.46 Let \( X_1, X_2, X_3, Y_1, Y_2, Y_3, Y_4 \) be i.i.d. \( N(\mu, \sigma^2) \) random variables. Let \( \bar{X} = \frac{1}{3} \sum_{i=1}^{3} X_i \) and \( \bar{Y} = \frac{1}{4} \sum_{j=1}^{4} Y_j \). If \( k = \sqrt{\frac{15}{7} \frac{(X - \bar{X})}{\sqrt{\sum_{i=1}^{3} (X_i - \bar{X})^2 + \sum_{j=1}^{4} (Y_j - \bar{Y})^2}}} \) has \( t_v \) distribution, then \( (v - k) \) equals __________
Q.47 Let \( f: \left[0, \frac{\pi}{2}\right] \to \mathbb{R} \) be defined as

\[
f(x) = ax + \beta \sin x,
\]

where \( a, \beta \in \mathbb{R} \). Let \( f \) have a local minimum at \( x = \frac{\pi}{4} \) with \( f\left(\frac{\pi}{4}\right) = \frac{\pi-4}{4\sqrt{2}} \).

Then \( 8\sqrt{2} \alpha + 4 \beta \) equals ________

Q.48 The area bounded between two parabolas \( y = x^2 + 4 \) and \( y = -x^2 + 6 \) is ________

Q.49 For \( j = 1, 2, \ldots, 5 \), let \( P_j \) be the matrix of order \( 5 \times 5 \) obtained by replacing the \( j^{th} \) column of the identity matrix of order \( 5 \times 5 \) with the column vector \( v = (5 \quad 4 \quad 3 \quad 2 \quad 1)^T \). Then the determinant of the matrix product \( P_1 P_2 P_3 P_4 P_5 \) is ________

Q.50 Let

\[
u_n = \frac{18n + 3}{(3n-1)^2(3n+2)^2}, \quad n \in \mathbb{N}.
\]

Then \( \sum_{n=1}^{\infty} u_n \) equals ________

Q.51 – Q. 60 carry two marks each.

Q.51 Let a unit vector \( v = (v_1 \quad v_2 \quad v_3)^T \) be such that \( Av = 0 \) where

\[
A = \begin{pmatrix}
5 & -1 & -1 \\
6 & 3 & 6 \\
-1 & 1 & 1 \\
3 & 3 & 3 \\
-1 & 1 & 5 \\
6 & 3 & 6
\end{pmatrix}.
\]

Then the value of \( \sqrt{6} \left(|v_1| + |v_2| + |v_3|\right) \) equals ________

Q.52 Let

\[
F(x) = \int_{0}^{x} e^{t(t^2 - 3t - 5)} dt, \quad x > 0.
\]

Then the number of roots of \( F(x) = 0 \) in the interval \( (0, 4) \) is ________
Q.53 A tangent is drawn on the curve $y = \frac{1}{3} \sqrt[3]{x^3}$, $(x > 0)$ at the point $P \left(1, \frac{1}{3}\right)$ which meets the x-axis at $Q$. Then the length of the closed curve $OQPO$, where $O$ is the origin, is

Q.54 The volume of the region
$$R = \{(x, y, z) \in \mathbb{R}^3 : x + y + z \leq 3, y^2 \leq 4x, 0 \leq x \leq 1, y \geq 0, z \geq 0\}$$
is

Q.55 Let $X$ be a continuous random variable with the probability density function
$$f(x) = \begin{cases} \frac{x}{8}, & \text{if } 0 < x < 2 \\ \frac{k}{8}, & \text{if } 2 \leq x \leq 4 \\ \frac{6-x}{8}, & \text{if } 4 < x < 6 \\ 0, & \text{otherwise.} \end{cases}$$

where $k$ is a real constant. Then $P(1 < X < 5)$ equals

Q.56 Let $X_1, X_2, X_3$ be independent random variables with the common probability density function
$$f(x) = \begin{cases} 2e^{-2x}, & \text{if } x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Let $Y = \min \{X_1, X_2, X_3\}, E(Y) = \mu_Y$ and $\text{Var}(Y) = \sigma_Y^2$. Then $P(\mu_Y < Y < \mu_Y + \sigma_Y)$ equals

Q.57 Let $X$ and $Y$ be continuous random variables with the joint probability density function
$$f(x, y) = \begin{cases} \frac{1}{2} e^{-x}, & \text{if } |y| \leq x, \ x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Then $E(X \mid Y = -1)$ equals

Q.58 Let $X$ and $Y$ be discrete random variables with $P(Y \in \{0, 1\}) = 1$,
$$P(X = 0) = \frac{3}{4}, \quad P(X = 1) = \frac{1}{4}, \quad P(Y = 1 \mid X = 1) = \frac{3}{4}, \quad P(Y = 0 \mid X = 0) = \frac{7}{8}.$$ 

Then $3P(Y = 1) - P(Y = 0)$ equals
Q.59
Let $X_1, X_2, \ldots, X_{100}$ be i.i.d. random variables with $E(X_1) = 0$, $E(X_1^2) = \sigma^2$, where $\sigma > 0$. Let $S = \sum_{i=1}^{100} X_i$. If an approximate value of $P(S \leq 30)$ is 0.9332, then $\sigma^2$ equals__________

Q.60 Let $X$ be a random variable with the probability density function

$$f(x|r, \lambda) = \frac{\lambda^r}{(r-1)!} x^{r-1}e^{-\lambda x}, \quad x > 0, \lambda > 0, r > 0.$$  

If $E(X) = 2$ and $\text{Var}(X) = 2$, then $P(X < 1)$ equals________________

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